Hwk 5 MAT 5190 M. ALVO
on 19

6.13 Let $Y_1 = \log X_1$, $Y_2 = \log X_2$. They are i.i.d. Their pdf is

$$f(y;\theta) = \theta \exp\left[\theta y - e^{-\theta y}\right]$$

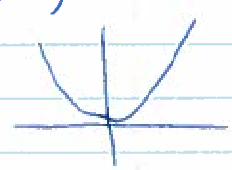
$$= \frac{1}{(1/\theta)} \exp\left[\frac{y}{(1/\theta)} - e^{-y/(1/\theta)}\right], \quad |y| < \infty \text{ which is}$$

(3) a scale family with parameter $1/\theta$.

Set $Z_i = \frac{1}{\theta} Y_i$ and hence $\frac{\log X_1}{\log X_2} = \frac{Y_1}{Y_2} = \frac{Z_1}{Z_2}$ indep of θ

$\therefore \frac{\log X_1}{\log X_2}$ is ancillary.

6.15 a) Let $\phi = a\theta^2$. The parameter space (θ, ϕ) consists of a quadratic and does not contain an open set.



Clearly $(\bar{X}, S^2)$ is sufficient and $E[S^2] = \phi$

(3) $E[\bar{X}^2] = Var[\bar{X}] + (E[\bar{X}])^2 = \frac{a\theta^2}{n} + \theta^2$. Hence

$$E\left[\frac{n}{a+n} \cdot \bar{X}^2 - \frac{S^2}{a}\right] = \frac{n}{a+n}\left[\frac{a\theta^2}{n} + \theta^2\right] - \frac{a\theta^2}{a}$$

$$= 0$$

yet $\frac{n}{a+n}\bar{X}^2 - \frac{S^2}{a} \neq 0$

So that $(\bar{X}, S^2)$ is not complete.

6.19 For distribution 1,

$$E_p[g(X)] = \sum_0^2 g(x) P(X=x) = p\,g(0) + 3p\,g(1) + (1-4p)g(2)$$

Let $g(0) = -3g(1)$ & $g(2) = 0$. Clearly $E_p[g(X)] = 0$ yet $g \neq 0$

(3) For distribution 2,

$$E_p[g(X)] = p\,g(0) + p^2 g(1) + (1-p-p^2)\,g(2)$$

$$= p^2(g(1)-g(2)) + (g(0)-g(2))p + g(2)$$

This is a polynomial of degree 2 in p. It is 0 for all p if and only if all the coefficients are 0. That is $g(1) = g(2) = g(0) = 0$.

6.23 The ratio $\dfrac{f(x;\theta)}{f(y;\theta)} = \dfrac{\theta^{-n} I_{(x_{(n)}/2,\, x_{(1)})}(\theta)}{\theta^{-n} I_{(y_{(n)}/2,\, y_{(1)})}(\theta)}$ is constant

if and only if $x_{(1)} = y_{(1)}$, $x_{(n)} = y_{(n)}$.

(3) If a function of the sufficient statistic is ancillary then the sufficient statistic is not complete.
Let $Z_1, \ldots, Z_n$ be a random sample from the uniform on $(1, 2)$.
Then $X_i = \theta Z_i$ is " " " " " " " $(\theta, 2\theta)$

$\therefore$ $X_{(1)}/X_{(n)} = Z_{(1)}/Z_{(n)}$ has a distribution independent of θ and hence is ancillary. However it is a function of the sufficient statistic and so the sufficient statistic is not complete

6.24

If $\lambda = 0$, $E[h(X)] = h(0) = 0$ implies $h(\cdot) = 0$. So completeness holds

(3) If $\lambda = 1$, $E[h(X)] = e^{-1}h(0) + e^{-1}\sum_{x=1}^{\infty} h(x)/x! = 0$ does not imply $h(x) = 0$ for all x. For example, $h(x) = (x-1)^2 - 1$ makes

$$E[h(X)] = E[X-1]^2 - 1 = \operatorname{Var} X - 1 = 1 - 1 = 0.$$

6.42 a) For the group G_1, to estimate θ we must have for

$\bar{g}_{a,c}(\theta) = c\theta + a$,

$$W(g(x_1 \ldots x_n)) = \bar{g}(W(x_1 \ldots x_n))$$

Now $W(g(x_1 \ldots x_n)) = W(cx_1 + a, \ldots, cx_n + a)$

whereas $\bar{g}(W(x_1 \ldots x_n)) = c\,W(x_1 \ldots x_n) + a$. Hence

$$(*) \qquad W(cx_1 + a, \ldots, cx_n + a) = c\,W(x_1, \ldots, x_n) + a.$$

Estimators of the form

$W(x_1 \ldots x_n) = \bar{x} + k$ would yield in $(*)$

(4) $LHS = \frac{1}{n}\Sigma(cx_i + a) + k = c\bar{x} + a + k$

$RHS = c(\bar{x} + k) + a = c\bar{x} + a + ck$ and so $(*)$ is not satisfied

On the other hand for the Group G_2, $\bar{g}_a(\theta) = \theta + a$ and we must have

$$(**) \qquad W(x_1 + a, \ldots, x_n + a) = W(x_1, \ldots, x_n) + a$$

Estimators of the form

$W(x_1 \ldots x_n) = \bar{x} + k$ would yield in $(**)$

$LHS = \frac{1}{n}(\Sigma(x_i + a)) + k = \bar{x} + a + k$

$RHS = \bar{x} + k + a$. and so $(**)$ is satisfied.

b) For the location scale family, X_i and $\sigma Z_i + \theta$ have the same distribution where $Z \sim f(x)$. (Theorem 3.5.6 p 120).

For G_1, set $c = \frac{1}{\sigma}$, $a = -\frac{\theta}{\sigma}$ in $(*)$. Then $(*)$ becomes

$E\,W(X_1 \ldots X_n) = \sigma E\,W(Z_1 \ldots Z_n) + \theta$ and we must have

$E\,W(Z_1 \ldots Z_n) = 0$

For G_2, set $a = -\theta$ and $(**)$ becomes $EW(\sigma Z_1, \ldots, \sigma Z_n) + \theta = EW(X_1 \ldots X_n)$

and we must have $EW(\sigma Z_1, \ldots, \sigma Z_n) = 0$